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THE ROLE OF CONCEPTUAL INTEGRATION AND SIMPLE DYNAMIC SCENARIOS IN THE MEANING CONSTRUCTION OF THE MAPPING IN MATHEMATICS

Abstract

Over the last two decades the impact of conceptual metaphor and conceptual blending on mathematics has been extensively researched (Lakoff & Núñez, 2000; Fauconnier & Turner, 2002; Turner, 2005; Núñez, 2006; Alexander, 2011; Turner, 2012; Danesi, 2016; Woźny, 2018). This paper examines the manner in which simple dynamic scenarios allow, through the process of conceptual integration, for multiple ways of constructing the meaning of a mathematical mapping. The paper analyses selected fragments extracted from two popular academic mathematics textbooks to ascertain how the authors use a number of simple dynamic scenarios to explain the concept. The paper then demonstrates how these dynamic scenarios help to avoid the problem of circularity of the (static) formal definition of the mapping. The results of the study indicate that conceptual blending may account for the flexibility of mathematics and its effectiveness in modelling the world around us.

Keywords: conceptual blending; mathematical mapping; embodied mathematics

1 Introduction

Mathematics continues to be productively applied in Engineering, Medicine, Chemistry, Biology, Physics, Social Sciences, communication and Computer Science, to name but a few. As Hohol (2011, p. 143) states, this fact is often treated by philosophers as an argument in favour of mathematical realism of the Platonian or Aristotelian variety. It is from this perspective that Quine-Putnam's "Indispensability Argument", Heller's "Hypothesis of the Mathematical Rationality of the World" or Tegmark's "Mathematical Universe Hypothesis" have been discussed. Eugene Wigner, a physicist who is often quoted in this context, ended his paper titled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" in the following manner:

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. (Wigner, 1960, p. 14)

James C. Alexander, a professor of mathematics, also views the “unreasonable effectiveness” of mathematics as a mystery but offers the following explanation for it:

It is a mystery to be explored that mathematics, in one sense a formal game based on a sparse foundation, does not become barren, but is ever more fecund. I posit, [...] that mathematics incorporates blending (and other cognitive processes) into its formal structure as a manifestation of human creativity melding into the disciplinary culture, and that features of blending, in particular emergent structure, are vital for the fecundity. (Alexander 2011, p. 3)

Over the last two decades the importance of conceptual blending and metaphor in mathematics has been extensively studied by, among others, Lakoff and Núñez (2000), Fauconnier and Turner (2002), Turner (2005), Núñez (2006), Alexander (2011), Turner (2012), Woźny, (2018). As an illustration, below are two quotations, starting with the ground-breaking *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being* by George Lakoff and Raphael Núñez.

Blends, metaphorical and nonmetaphorical, occur throughout mathematics. Many of the most important ideas in mathematics are metaphorical conceptual blends (2000, p. 48)

Mark Turner adds the concept of “small spatial stories” as a vital component of conceptual blending in mathematics:

Our advanced abilities for mathematics are based in part on our prior cognitive ability for story [...] — understanding the world and our agency in it through certain kinds of human-scale conceptual organizations involving agents and actions in space. Another basic human cognitive operation that makes it possible for us to invent mathematical concepts [...] is “conceptual integration,” also called “blending”. Story and blending work as a team. (2005, p. 4)

Small spatial stories are dynamic scenarios which constitute one of the inputs of the conceptual integration network, and they always involve agents/actors moving and manipulating (interacting with) objects. For example, a person moving objects from one place to another. The main claim of this article is twofold. Firstly, the crucial notion of mathematical mapping is “understood through” a number of selected small spatial stories, such as the one above. To be more precise, the algebra handbooks prompt the reader to understand mathematical mappings (functions) in this way. Secondly, incorporating small spatial stories — through conceptual blending — into the structure of mathematics is responsible for the effectiveness and productivity of the latter. To paraphrase Wigner from the quotation above, conceptual blending makes the effectiveness of mathematics “reasonable”.

The analysis presented in this paper does not concern the exact construction of conceptual integration networks. Instead, the primary focus is on the role of dynamic input scenarios and the number of their possible combinations (blends). In fact, for this reason, it would have been possible to build an argument on the simpler two-space mapping framework of Conceptual Metaphor Theory (CMT); however, Conceptual Blending Theory (CBT) was selected as the theoretical basis for two reasons. Firstly, CMT and CBT are not contradictory but complementary (Grady, Oakley, & Coulson, 1999). Secondly, in CBT the inter-space mapping can be bi-directional (Coulson, 1997;

Fauconnier & Turner, 2002), which seems to be the case, as will be demonstrated below when the focus turns to specific blends.

The next section will analyse fragments of two popular, academic level mathematics textbooks to see how mapping is defined and described. The choice of textbooks was not random – the aim was to find a standard, yet “chatty” textbook — which would allow easier access to the “mathematical mind”. The first of them is Nathaniel Herstein’s (1975) excellent *Topics in Algebra* — a classic textbook¹ addressed to “the most gifted sophomores in mathematics at Cornell” (p. 8). The second is *First Year Calculus For Students of Mathematics and Related Disciplines* by Michael M. Dougherty and John Gieringer (2012). Probably the only advanced calculus textbook which “is meant to be read, perhaps even curled up with and read” (p. iii).

2 The concept of mathematical mapping and small spatial stories

Nathaniel Herstein, in his popular *Topics in Algebra*, introduces mapping as “probably the single most important and universal notion that runs through all of mathematics” (Herstein, 1975, p. 10) Quite typically,² he defines it as follows:

If S and T are nonempty sets, then a mapping from S to T is a subset, M , of $S \times T$ such that for every s in S there is a unique t in T such that the ordered pair (s, t) is in M . (Herstein, 1975, p. 10)

He then adds:

This definition serves to make the concept of a mapping precise for us **but we shall almost never use it in this form. Instead we do prefer to think of a mapping as a rule which associates** (emphasis mine) with any element s in S some element t in T . (Herstein, 1975, p. 10)

A “precise” definition, yet one — for some reason — to be avoided. Dougherty and Gieringer (2012) in another popular textbook, and speaking of the same definition, express a similar reluctance:

while we mention it, **we will not use it within the rest of the text because it de-emphasizes functions as actions or processes** (emphasis mine) taking inputs and deterministically returning outputs”. (Dougherty & Gieringer, 2012, p. 124)

Both textbooks recommend understanding functions in terms of small spatial stories — dynamic scenarios in which objects are manipulated by agents. In the first there is “a rule” which performs the action of tying objects into pairs. In the second there is a black box which takes arguments and returns values. Herstein prompts for yet another small spatial story which can be used to construct the meaning of a mathematical mapping:

It is hardly a new thing to any of us, for we have been considering mappings from the very earliest days of our mathematical training. When we were asked to plot the relation $y = x^2$ we were simply being asked to study the particular **mapping which takes every real number onto its square** (emphasis mine). (Herstein, 1975, p. 10)

In this scenario the function becomes an actor who carries objects (numbers) from one place to another. So far we have had as many as three small spatial stories which are to be used as the basis

¹Undergraduate modern algebra courses are sometimes referred to as “Herstein level courses”.

²This definition can be found in all advanced level algebra textbooks and is usually referred to as the definition “by graph” or “Peano’s definition”.

of an understanding of mathematical mapping. Let us call them *the matchmaker*, *the black box*, and *the carrier* respectively. The matchmaker associates clients into pairs, the black box grinds the rough material of the arguments into the finished product of the function values, and the carrier moves cargo from the function domain to the target set. In Fig. 1 the black box scenario is diagrammed on the left. The diagram on the right represents both the matchmaker and the carrier.

Figure 1: Functions (mappings) as actions (Dougherty & Gieringer, 2012).

It may be surprising to see that the narrative of modern algebra prompts the reader to understand the crucial concept of mapping in so many ways. It even seems to undermine the status of mathematics as a paragon of formal rigour and precision. However, this polysemy may account for the fecundity of mathematics — it prevents mathematics from becoming barren (cf. Alexander, 2011, p. 3, quoted above).

To explain further, let us return to the static definition of mapping as a set of ordered pairs. We will try to see more clearly why both handbooks mentioned above³ avoid using it and instead prefer the dynamic scenarios of the black box, the matchmaker and the carrier. The reason is certainly not didactic, as the static definition seems simple enough: a mapping from set S to set T is a set of ordered pairs (s, t) so that every element s in S is paired with some element t in T . The “ordered pair” is one of the undefined primary concepts⁴ (just like set and element, for example). Let us briefly concentrate on the notion of an ordered pair. What makes a pair “ordered”? To answer this it is necessary to know which element comes first (or left) and which comes second (or right). In other words — we need a set of two indexes like 1, 2 or left, right or first, second, etc. and then we have to associate each of the indexes with the elements of the pair. Therefore, for example, “first” can be associated with x and “second” with y and thus an ordered pair of (x, y) is created. An ordered pair, therefore, is a mapping from the set of indexes to the set of the elements of the pair. This would certainly be a good definition but the static definition of a mapping (a set of ordered pairs) would then be circular-mapping would be defined as a set of mappings. This circularity is avoided by treating the ordered pair as a primary, undefined concept. The circularity is avoided formally but certainly not conceptually. To cite Alexander (2011, p. 3) once more, this definition of mapping is “barren” and implicitly circular.

If that is the case, then perhaps all the other ways of understanding mapping (the black box, the matchmaker, the carrier) are also implicitly circular. The carrier, for example, carries x onto

³Herstein (1975) and Dougherty and Gieringer (2012).

⁴Herstein (1975) does not define it and neither did Frege (1879).

x^2 and thus also creates ordered pairs of (2, 4), (3, 9), (4, 16) etc. How do we know that these pairs are ordered? We do not need any primary concepts — the order is built into the dynamic scenario. The number on the left is the one that is being carried and the number on the right indicates the spot on which the carried number is placed. The pairs are ordered because their left and right elements occupy different roles in the scenario. The same applies to the two other scenarios — the black box and the matchmaker. First, x is put into the black box, and then it returns $f(x)$. In the matchmaker scenario — x is the client for which a match y has to be found. The matchmaker has to know x first to find a suitable match y later. Because the scenarios are dynamic, the pairs (x, y) are ordered not only according to their roles, but also temporarily — x comes sooner, y comes later.

So far, as many as four ways of constructing the meaning of mapping have been mentioned:

1. *the set of ordered pairs* (static, circular)
2. *the matchmaker* (dynamic scenario)
3. *the black box* (dynamic scenario)
4. *the carrier* (dynamic scenario)

However, there are in fact many more and this is where conceptual blending comes into play:

There are many ways to describe functions. To name a couple, we can look at them as mappings (which “map” the independent variable values to their respective dependent variable outputs), and they can be described as processes or “machines”. We will include the abstract definition^a later, to be complete. **Ideally it is best to consider functions in all these ways** (emphasis JW). However for our purposes we will concentrate on the notion of functions as machines. (Dougherty & Gieringer, 2012, p. 124)

^aThe static one, a set of ordered pairs.

The reader of an algebra textbook is invited to understand mathematical mapping “in all these ways”. In other words, the reader is invited to blend (people tend to do so constantly, even without invitation). Each of the four construals above can be blended with the other three, which produces as many as six⁵ additional ways of constructing the meaning of this “most important and universal notion that runs through all of mathematics” (Herstein, 1975, p. 10). If these new six are added to the previous (“unblended”) four, there exist ten ways of constructing the meaning of mapping in mathematics. For example, in the blend of 1 and 2 one can imagine a set of pairs that were created by the matchmaker. In the blend of 2 and 3 we can imagine a matchmaker that has a machine⁶ for finding a suitable match, etc. It is worth noting at this point that the mappings between the scenarios are bi-directional. The pairs are created by a matchmaker but, conversely, the matchmaker may be matching their clients using a notebook containing (or a computer pre-programmed with) a set of ordered pairs, etc.

Of course, ten ways of understanding functions (mappings) is certainly an underestimate because an allowance has to be made for secondary blends (blends becoming inputs into other blends) and for other dynamic scenarios, which have not been discussed yet, that can be found in the algebra textbooks. For example, a graphical representation, which in fact is also a dynamic scenario as we know how a graph “works”: if one wants to know what the value of the function (mapping) for x is, they draw a line perpendicular to the x -axis until it crosses the graph and then from that point they draw another line parallel to the x -axis until it crosses the y -axis — this is where y is. We can call this scenario *the hiker* (Fig. 2) as it involves motion (trajectories) determined by signposts (the points of the function graph), which indicate where to turn.

⁵ $(4 * 3)/2 = 6$ (because the order of the input spaces does not matter).

⁶A computer with professional matchmaking software.

Figure 2: Graph of a function — the dynamic scenario of *the hiker*.

Yet another way of representing functions dynamically is with an expression, such as $f(x) = x^2 + 1$, which “describes the action (emphasis JW) of the function [...] the function in the above example can be described as a process, by which the input is first squared, and the result is added to 1” (Dougherty & Gieringer, 2012, p. 126). Let us call this dynamic scenario *the chef* (the function formula as a recipe). If these two ways of “understanding”⁷ mappings were added to the initial four, the extended list of input spaces would be as follows:

1. *the set of ordered pairs* (static, circular)
2. *the matchmaker* (dynamic scenario)
3. *the black box* (dynamic scenario)
4. *the carrier* (dynamic scenario)
5. *the hiker* (dynamic scenario)
6. *the chef* (dynamic scenario)

Again, each of these now six inputs can be blended with the other five, which increases the number of possible (first-order) blends to fifteen⁸. If these blends are added to the original “unblended” six from the list above there are twenty-one, which justifies the title of this paper. For example, the blend of 3 (the black box) and 6 (the processor) could result in a “transparent meat grinder” — a black box that we can look into in order to view the whole process of turning input into output. Blending 1 (the set of pairs) and 6 (the chef) would result in a set of pairs in which the right element is obtained by processing the left element. In the blended space of 3 (the black box) and 5 (the hiker) the hiker has to find their way from the input to the output of the black box. Again, it must be pointed out that twenty-one is an underestimate because of the iterative nature of the conceptual integration process (blends can be used as inputs again).

3 Summary and conclusion

It has been demonstrated how simple dynamic scenarios (small spatial stories) allow, through the process of conceptual integration, for multiple ways of constructing the meaning of mathematical mapping. Fragments of two popular mathematics textbooks were analysed to find that the authors suggest the use of small spatial stories, which were labelled *the matchmaker*, *the black box*, *the carrier*, *the hiker* and *the chef*. It has also been shown how these dynamic scenarios help to

⁷i.e. constructing conceptual integration networks with those dynamic scenarios as inputs.

⁸Each of the six inputs can be paired with the other five, which gives us $6 * 5 = 30$ combinations but, since the order of blending does not matter, we have to divide 30 by 2, with the final result of 15.

avoid the problem of circularity (“barrenness”) of the static, formal definition of mapping as a set of ordered pairs (a.k.a. “definition by graph” or “definition of Peano”). Both textbooks quoted (Herstein, 1975, Dougherty & Gieringer, 2012) include this definition and describe it as “precise” or “abstract” respectively but at the same time advise the reader to understand mapping in terms of dynamic scenarios (“small spatial stories”) in which actors manipulate objects. One of the reasons for selecting this strategy may be the implicit circularity of the formal definition, which is based on the undefined notion of an “ordered pair”. The dynamic scenarios do not require the artificial introduction of “order” as a primary concept because they are “naturally ordered” in two ways. Firstly, the argument and value occupy different roles in the scenarios. Secondly, the x, y sets are ordered according to the timeline — in the scenarios x always comes before y . It has been shown that the crucial concept of order enters the foundations of mathematics via a number of dynamic scenarios. A more general conclusion is that the flexibility of mathematics, its ability to keep pace with the rapid development of technology and natural sciences may, at least in part, stem from the polysemy of crucial mathematical concepts. This polysemy is based on the inventory of “small spatial stories”, which can be selected as inputs for multiple conceptual integration networks and which, in a sense, “provide what is missing”, enriching the rigid and potentially barren, set-theoretical foundation of mathematics.

References

- Alexander, J. (2011). Blending in mathematics. *Semiotica*, 2011(187), 1–48. <https://doi.org/10.1515/semi.2011.063>
- Coulson, S. (1997). *Semantic leaps: The role of frame-shifting and conceptual blending in meaning construction* (Ph.D. Dissertation). University of California, San Diego.
- Danesi, M. (2016). *Language and mathematics: An interdisciplinary guide*. New York, NY: Mouton de Gruyter. <https://doi.org/10.1515/9781614513186>
- Dougherty, M., & Gieringer, J. (2012). *First year calculus for students of mathematics and related disciplines*. Retrieved December 11, 2017, from <http://faculty.swosu.edu/michael.dougherty/book/book.pdf>
- Doxiadis, A., & Mazur, B. (Eds.) (2011). *Circles disturbed: The interplay of mathematics and narrative*. Princeton, NJ: Princeton University Press.
- Fauconnier, M., & Turner, M. (2002). *The way we think: Conceptual blending and the mind’s hidden complexities*. New York, NY: Basic Books.
- Frege, G. (1879). *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought*. Retrieved December 28, 2017, from <http://dec59.ruk.cuni.cz/~kolmanv/Begriffsschrift.pdf>
- Grady, J., Oakley, T., & Coulson, S. 1999. Blending and metaphor. In G. Steen, & R. Gibbs (Eds.), *Metaphor in cognitive linguistics* (pp. 101–124). Philadelphia, PA: John Benjamins.
- Hausdorff, F. (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit.
- Herstein, I. (1975). *Topics in algebra*. New York, NY: John Wiley & Sons.
- Hohol, M. (2011). Matematyczność ucieleśniona. In B. Brożek, J. Mączka, W. P. Grygiel, & M. Hohol (Eds.), *Oblicza racjonalności: Wokół myśli Michała Hellera* (pp. 143–166). Kraków: Copernicus Center Press.
- Lakoff, G. (1987). *Women, fire and dangerous things: What categories reveal about the mind*. Chicago, IL: Chicago University Press.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.
- Núñez, R. (2006). Do real numbers really move?. In R. Hersh (Ed.), *18 unconventional essays on the nature of mathematics* (pp. 160–181). New York, NY: Springer.
- Turner, M. (2005). *Mathematics and narrative*. Paper presented at the International Conference on Mathematics and Narrative, Mykonos, Greece, 12–15 July 2005. http://thalesandfriends.org/wp-content/uploads/2012/03/turner_paper.pdf
- Turner, M. (2012). Mental packing and unpacking in mathematics. In M. Bockarova, M. Danesi, & R. Núñez (Eds.), *Semiotic and Cognitive Science Articles on the Nature of Mathematics* (pp. 248–267). Munich: Lincom Europa.

Wigner, E. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications in Pure and Applied Mathematics*, 13(1), 1–14. <https://doi.org/10.1002/cpa.3160130102>

Woźny, J. (2018). *How we understand mathematics. conceptual integration in the language of mathematical description*. New York, NY: Springer. <https://doi.org/10.1007/978-3-319-77688-0>

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